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This Journal is dedicated to the following aims:

1. Through published standard papers on the culture aspects, humanism and history of mathematics to deepen and to widen public interest in its values.
2. To supply an additional medium for the publication of expository mathematics.
3. To promote more scientific methods of teaching mathematics.
4. To publish and to distribute to groups most interested high-class papers of research quality representing all mathematical fields.

Louisiana State University Host to 1940 December Meetings of A. M. S. and M. A. of A.

Gratifying to mathematical workers throughout the Southland will be these official communications from the Secretaries of The Mathematical Association of America and the American Mathematical Society. On April 14 from Secretary R. G. D. Richardson of the American Mathematical Society came a letter, the opening paragraph of which was as follows: "At the recent meeting of the Council of the Society, the report of the Committee on the Place of the 1940 Annual Meeting was presented and adopted as per the enclosed copy,

> '1. We recommend that the invitation of Louisiana State University, at Baton Rouge, be accepted. . . . . .
>
> . . . . . .
>
> Signed: Philip Franklin
> E. T. Browne
> W. L. Ayres, Chairman'."

On May 23 came a letter from Secretary W. D. Cairns of The Mathematical Association of America, reading in part as follows: "You will be glad to learn that the Trustees of the Mathematical Association have voted in favor of meeting with you in December, 1940. I have only recently been able to bring this to the Trustees and only now has the vote been complete."

It is difficult to conceive a more wholesome type of stimulation to mathematical activity in the far South than will be this gathering in Baton Rouge of the members of the two greatest organizations of teachers and researchers in the field of mathematics in America. Anticipation of such a gathering should be inwrought with a sense of challenge and a conviction of opportunity.

S. T. Sanders.

Concerning the Distribution of the Mean of r Independent Chance Variables When Each Is Subject to the Frequency Law

$$\frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)}\, x^{p-1}(1-x)^{q-1}$$

By William Dowell Baten
Michigan State College

Let the probability of selecting the chance real variable x from the interval $(x,\ x+dx)$, be to within infinitesimals of higher order, the quantity

$$f(x)dx=\frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)}\, x^{p-1}(1-x)^{q-1}dx, \tag{1}$$

where x runs from 0 to 1 and p and q are positive quantities. The object of this article is to find, by using Pearson's Type criteria, the Pearson Type of the probability function for the mean

$$z=\sum_{i=1}^{r} x_i/r,$$

when each variable x_i is subject to the probability function (1).

The first four moments about the mean of $f(x)$ are:

$$\mu_{1:x}=0; \qquad \mu_{2:x}=\frac{pq}{(p+q)(p+q+1)^{(2)}}\ ;*$$

$$\mu_{3:x}=\frac{2pq(q-p)}{(p+q)^3(p+q+2)^{(2)}}\ ;*$$

$$\mu_{4:x}=\frac{3pq}{(p+q)^4(p+q+3)^{(3)}}\,(p^2q+pq^2+2p^2-2pq+2q^2).$$

*$w^{(k)}=w(w-1)(w-2)\ldots(w-k+1)$.

According to the theory of sampling from r parent populations,[a] the first four months about the mean of the distribution of the mean are (where only one variable is taken from each parent) as follows:

$$\mu_{1:s}=0;\quad \mu_{2:s}=\frac{\mu_{2:x}}{r}=\frac{pq}{r(p+q)^2(p+q+1)};$$

$$\mu_{3:s}=\frac{\mu_{3:x}}{r^2}=\frac{2pq(q-p)}{r^2(p+q)^3(p+q+2)^{(2)}};$$

$$\mu_{4:s}=\frac{1}{r^3}[\mu_{4:x}+3(r-1)\mu^2{}_{2:x}]$$

$$=\frac{\left(\begin{array}{c}3pq[p^3(rq+2)+q^3(rp+2)+pq(p+q)(5r-4)\\+2(p-q)^2+2pq(3r-2)+2rp^2q^2]\end{array}\right)}{r^3(p+q)^4(p+q+1)^2(p+q+3)^{(2)}}.$$

Let us now use Pearson's criteria for determining the Type to which the law of the mean belongs, namely[c]

$$K=\frac{\beta_{1:s}(\beta_{2:s}+3)^2}{4(2\beta_{2:s}-3\beta_{1:s}-6)(4\beta_{2:s}-3\beta_{1:s})},$$

where

$$\beta_{1:s}=\frac{\mu_{3:s}{}^2}{(\mu_{2:s})^3}=\frac{4(p-q)^2(p+q+1)}{rpq(p+q+2)^2},$$

and

$$\beta_{2:s}=\frac{\mu_{4:s}}{(\mu_{2:s})^2}=\frac{\left(\begin{array}{c}3[rpq(p+q+3)^{(2)}+2(p+q)^2(p+q+1)\\-2pq(5p+5q+6)]\end{array}\right)}{rpq(p+q+3)^{(2)}}$$

$$\beta_{2:s}+3=\frac{6[(p+q)^2(rpq+p+q+1)+pq(5p+5q+6)(r-1)]}{rpq(p+q+3)^{(2)}};$$

$$2\beta_{2:s}-3\beta_{1:s}-6=-\frac{12(p+q)^2(p+1)(q+1)}{rpq(p+q+3)^{(2)}(p+q+2)};$$

$$4\beta_{2:s}-3\beta_{1:s}=\frac{\left(\begin{array}{c}12\{(p+q)^2[(p+q)(rpq+2+p+q)+pq(7r-6)+1]\\+4pq(r-1)(4p+4q+3)\}\end{array}\right)}{rpq(p+q+2)^2(p+q+3)}.$$

Therefore

$$K=-\frac{\left\{\begin{array}{c}(q-p)^2(p+q+1)[(p+q)^2(rpq+p+q+1)\\+(pq(5p+5q+6)(r-1)]^2\end{array}\right\}}{\left\{\begin{array}{c}4rpq(p+q)^2(p+1)(q+1)\{(p+q)^2[(p+q)(rpq+p+q+2)\\+pq(7r-6)+1]+4pq(r-1)(4p+4q+3)\}\end{array}\right\}}.$$

The numerator and denominator of K are always positive if p and q are greater than zero and r is greater than or equal to 1; under these conditions K is always negative. Therefore according to Pearson's criteria the distribution of the mean of r chance variables is a Pearson Type I distribution.

Student used this method for finding the probable error of the mean.[d] Church studied this problem and found several Types for the mean when sampling from various parents.[e]

If the distribution of the means really is a Type I distribution its moments about the mean should satisfy the recursion formula for moments pertaining to this type. Type I arises from the differential equation (x is now measured from the mean).

$$\frac{dy}{y}=\frac{(a-x)dx}{b+cx+dx^2}, \tag{2}$$

where y represents the frequency law for the chance variable x and a,b,c, and d are constants. By multiplying (2) by x^n and integrating between L_1 and L_2, at which y and $x^k y$ are equal to zero, one obtains the following recursion formula for the moments

$$n\mu_{n-1:x}\cdot b+(n+1)\mu_{n:x}\cdot c+(n+2)\mu_{n+1:x}\cdot d=\mu_{n+1:x}-\mu_{n:x}\cdot a. \tag{3}$$

Letting $n=0,1,2,3$, one secures four equations, from which the following values of a, b, c, and d are obtained.

$$a=\frac{-\mu_{3:x}[\mu_{4:x}+3\mu_{2:x}^2]}{10\mu_{2:x}\mu_{4:x}-18\mu_{2:x}^3-12\mu_{3:x}^2};\qquad b=\frac{\mu_{2:x}[4\mu_{2:x}\mu_{4:x}-3\mu_{3:x}^2]}{10\mu_{2:x}\mu_{4:x}-18\mu_{2:x}^3-12\mu_{3:x}^2}$$

$$c=\frac{\mu_{3:x}[\mu_{4:x}+3\mu_{2:x}^2]}{10\mu_{2:x}\mu_{4:x}-18\mu_{2:x}^3-12\mu_{3:x}^2};\qquad d=\frac{2\mu_{2:x}\mu_{4:x}-6\mu_{2:x}^3-3\mu_{3:x}^2}{10\mu_{2:x}\mu_{4:x}-18\mu_{2:x}^3-12\mu_{3:x}^2}.$$

Substituting these values of a, b, c, and d in (3) gives the recursion formula for moments about the mean for $f(x)$

$$\begin{aligned}n\mu_{2:x}[4\mu_{2:x}\mu_{4:x}-3\mu_{3:x}^2]\mu_{n-1:x}+n\mu_{3:x}[\mu_{4:x}+3\mu_{2:x}^2]\mu_{n:x}\\-[2(3-n)\mu_{2:x}\mu_{4:x}+6(n-1)\mu_{2:x}^3+3(n-2)\mu_{3:x}^2]\mu_{n+1:x}=0.\end{aligned} \tag{4}$$

If the distribution of the sample averages is a Type I distribution its moments about the mean should satisfy (4) where the moments are the moments of the distribution of the mean, or

$$(5)\quad n\mu_{2:z}[4\mu_{2:z}\mu_{4:z}-3\mu_{3:z}^2]\mu_{(n-1):z}+n\mu_{3:z}[\mu_{4:z}+3\mu_{2:z}^2]\mu_{n:z} -[2(3-n)\mu_{2:z}\mu_{4:z}+6(n-1)\mu_{2:z}^3+3(n-2)\mu_{3:z}^2]\mu_{n+1:z}=0,$$

for all values of n. Equation (5) is satisfied when $n=0,1,2,3$ for the Pearson criterion for his Types involve the first four moments.

Let us consider (5) when $n=4$; it becomes after simplifying

$$(6)\quad (4\mu_{2:z}\mu_{4:z}-3\mu_{3:z}^2)(2\mu_{2:z}\mu_{3:z}+\mu_{5:z}) +(\mu_{4:z}+3\mu_{2:z}^2)(2\mu_{3:z}\mu_{4:z}-3\mu_{2:z}\mu_{5:z})=0.$$

The fifth moment about the mean of (1) is

$$\mu_{5:x}=\frac{4pq(q-p)(6q^2+6p^2+5pq^2+5p^2q)}{(p+q)^5(p+q+4)^{(4)}},$$

and the fifth moment about the mean of the distribution of the means is

$$\mu_{5:z}=\frac{1}{r^4}[\mu_{5:x}+10(r-1)\mu_{2:x}\mu_{3:x}].^{(a),\,(b)}$$

By using the values found above the factors of (6) are found to be

$$4\mu_{2:z}\mu_{4:z}-3\mu_{3:z}^2 =\frac{\left\{\begin{array}{c}12p^2q^2[(p+q)^2(p+1)(q+1)(p+q+1)\\ +pq(r-1)(p+q+2)^2(p+q+3)]\end{array}\right\}}{r^4(p+q)^6(p+q+1)^3(p+q+2)^2(p+q+3)};$$

$$2\mu_{2:z}\mu_{3:z}+\mu_{5:z} =\frac{\left\{\begin{array}{c}24pq(q-p)[p^3+q^3-4p^2q-4p\cdot q^2+p^2+q^2\\ -10pq+rpq(p+q+4)^{(2)}]\end{array}\right\}}{r^4(p+q)^5(p+q+1)^2(p+q+2)(p+q+4)^{(2)}};$$

$$\mu_{4:z}+3\mu_{2:z}^2 =\frac{\left\{\begin{array}{c}6pq[(p^3+q^3-2pq^2-2p^2q+p^2+q^2-4pq)\\ +rpq(p+q+3)^{(2)}]\end{array}\right\}}{r^3(p+q)^4(p+q+1)^2(p+q+3)^{(2)}};$$

$$2\mu_{3:z}\mu_{4:z}-3\mu_{2:z}\mu_{5:z} =-\frac{\left\{\begin{array}{c}48p^2q^2(q-p)[(p+q)^2(p+1)(q+1)(p+q+1)\\ +pq(r-1)(p+q+4)^{(3)}\end{array}\right\}}{r^5(p+q)^7(p+q+2)^{(2)}(p+q+4)^{(4)}(p+q+1)}$$

When values of these factors are substituted in (6) the recursion formula for the moments of the distribution of the mean reduces to

$$r = 1,$$

which means that (6) is true when there is one and only one item in each sample or when one samples from one distribution with one item in the sample. In this case the distribution of the means is the original parent (1). Hence for samples containing more than one item the law of the mean, in the case under consideration, is not a Type I distribution, although its first four moments do satisfy the Pearson criterion for a Type I frequency distribution.

The above has shown that a distribution may satisfy the Pearson criterion for Type I without being a Type I distribution.

Irwin found the distribution of the means when sampling from a Type I parent.[f]

LITERATURE

[a]This might be obtained by sampling n times from the same parent, see Carver, H. C. "Editorial", The Annals of Math. Statistics, Vol. 1, 1930, pp. 260-274.

[b]Baten, W. D. "Sampling from many parent populations" The Tôhoku Math. Jour., Vol. 36, Part II. (1933) pp. 206-222.

[c]Tables for Statisticians, Vol. I, pp. lxiii.

[d]"Student", "The probable error of a mean", Biometrika, Vol. VI, pp. 1-25.

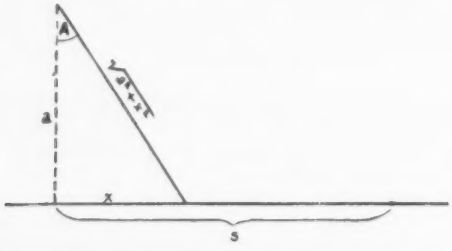
[e]Church, A. E. R., "On the means and squared standard-deviations of small samples from any population." Biometrika, Vol. 18, 1926, pp. 221-394.

[f]Irwin, J. O., Metron, 1930.

A Problem in Minimum Values

By F. A. Rickey
Louisiana State University

In many calculus texts can be found a problem stated essentially as follows: "A man is in a boat 5 miles from a straight shore line. He can row at the rate of 3 miles per hour and walk at the rate of 4 miles per hour. How far from the point on shore directly opposite him should he land in order to reach a point s miles down the shore in the shortest time?"



The solution is usually obtained by expressing the time t as a function of the required distance x. Thus

$$t=t(x)=\frac{\sqrt{x^2+25}}{3}+\frac{s-x}{4}.$$

From which,
$$t'(x)=\frac{x}{3\sqrt{x^2+25}}-\tfrac{1}{4}$$

and
$$t''(x)=\frac{25}{3(x^2+25)^{3/2}}.$$

The positive root of $t'(x)=0$ is $x=15/\sqrt{7}=$appr. 5.67 miles. Since $t'(15/\sqrt{7})=0$ and $t''(15/\sqrt{7})>0$, the conclusion is given that the time is a minimum for $x=5.67$ miles.

This solution obviously fails if s is less than $15/\sqrt{7}$, for certainly the boatsman would not row *past* his destination and then walk back to it in order to economize his time. Yet we have reason to believe that there must be a point at which he should land which should produce a trip of minimum time for every positive value of s.

The correct expression for the $t(x)$ of this problem as given below is typical of the class of functions which have a maximum or minimum value, but which do not necessarily have a vanishing first derivative.* Calculus students, in being introduced to the subject, are shown graphically that such functions may exist, but this possibility seems to be relegated to the background when *applied* problems are proposed. We offer the following solution and discussion of the rowing problem as an interesting example of the class of functions just mentioned.

Let us first re-state the problem in the more general form: "A man in a boat is a miles from a straight shore line and wishes to reach a point s miles downshore. If he can row r miles per hour and walk w miles per hour, how far downshore should he land in order to reach his destination in a minimum of time?"

We express the time t as a function $t(x)$ of the required distance x thus,†

$$t(x)=\frac{\sqrt{x^2+a^2}}{r}+\frac{\sqrt{(s-x)^2}}{w}.$$

Hence

$$(1)\qquad t'(x)=\frac{x}{r\sqrt{x^2+a^2}}-\frac{s-x}{w\sqrt{(s-x)^2}}$$

$$=\frac{x}{r\sqrt{x^2+a^2}}-\varphi(s-x),$$

where $\varphi(s-x)=1/w$ for $x<s$

$=-1/w$ for $x>s$ and is not defined for $x=s$.

The expression for $t''(x)$ readily reduces to

$$t''(x)=\frac{a^2}{r(x^2+a^2)^{3/2}}>0 \qquad (x\neq s).$$

Following the usual method for determining maximum or minimum values of a function, we find the possible positive root of $t'(x)=0$

*Cf. de la Vallée Poussin, *Cours d'Analyse Infinitésimale*, Vol. I, 3rd ed. Page 132, for a clear discussion.

†We note that the failure of the method used in solving the original problem was due to the use of the expression $s-x$ for the distance walked on the shore. Obviously, the expression $|s-x|$ or $\sqrt{(s-x)^2}$ is necessary if the time is to be considered for $x>s$.

to be $x=ar/\sqrt{w^2-r^2}$. However, upon substitution in the expression for $t'(x)$, we find after reduction that

$$t'(ra/\sqrt{w^2-r^2}) = \frac{1}{w} - \varphi(s-ar/\sqrt{w^2-r^2})$$
$$= 0 \quad \text{if } s > ar/\sqrt{w^2-r^2},$$
$$= 2/w \text{ if } s < ar/\sqrt{w^2-r^2},$$

and is not defined if $s=ar/\sqrt{w^2-r^2}$.

We thus divide our problem into the following cases:

Case 1, $s>ar/\sqrt{w^2-r^2}$. This is the ordinary case, in which the function $t(x)$ is continuous in x, $t'(x)$ vanishes for $x=ar/\sqrt{w^2-r^2}$, and $t''(ar/\sqrt{w^2-r^2})>0$. Hence the time t has a minimum value for $x=ar/\sqrt{w^2-r^2}$.

Case 2, $s\leqq ar/\sqrt{w^2-r^2}$. Using the notation

$$\Theta(x) = \frac{x}{r\sqrt{x^2+a^2}},$$

we may write

$$t'(x) = \Theta(x) - \varphi(s-x). \tag{2}$$

Since our discussion is restricted to positive values of x, we see that $\Theta(x)$ can be written

$$\Theta(x) = \frac{1}{r\sqrt{1+a^2/x^2}},$$

from which it is obvious that $\Theta(x)$ *increases with* x. Furthermore, we have seen by substitution that $\Theta(ar/\sqrt{w^2-r^2}) = 1/w$. From these two facts it follows that

$$\Theta(s-h)<1/w, \quad (0<s\leqq ar/\sqrt{w^2-r^2}, \quad 0<h<s) \tag{3}$$
$$\Theta(s+h)>0, \quad (0<s\leqq ar/\sqrt{w^2-r^2}, \quad 0<h).$$

Hence from (1), (2), and (3), it follows that for the given case

$$t'(s-h) < \frac{1}{w} - \frac{1}{w} = 0$$

and

$$t'(s+h) > 0 + \frac{1}{w} > 0.$$

We now have the following pertinent information as to $t(x)$ and its first and second derivatives for the case $s \leqq ar/\sqrt{w^2-r^2}$:

$t(x)$ is a continuous function of x.

$t'(s-h)<0, \quad t'(s+h)>0.$

$t''(x)>0, \quad (x \neq s).$

From this we have the conclusion that if $s \leqq ar/\sqrt{w^2-r^2}$, $t=t(x)$ is minimum for and only for $x=s$.

Alternate statement of the conclusions. From the above figure, in which the angle A is the angle formed by the line of direction of rowing and the normal to the shore line, we easily verify that for $s>ar/\sqrt{w^2-r^2}$, t is a minimum for $A=\sin^{-1}(r/w)$.* Hence we can state our con-clu sion in the form: "A man who can walk faster than he can row and who wishes to reach a point s miles downshore in minimum time should row at an angle of $\sin^{-1}(r/w)$ with the normal to the shore line, *provided* this does not cause him to row past his objective. Otherwise, he should row straight to his destination. If his rowing speed is not less than his walking speed, he should row straight to his destination."

Graphical representation of $t(s,x)$. The number s may be looked upon as the parameter of a family of curves each of which represents the values of t corresponding to varying values of x. The accompanying illustration shows some of the members of the family with $r=2$, $w=3$, and $a=4$.

It is interesting to note that if we eliminate s from the equation

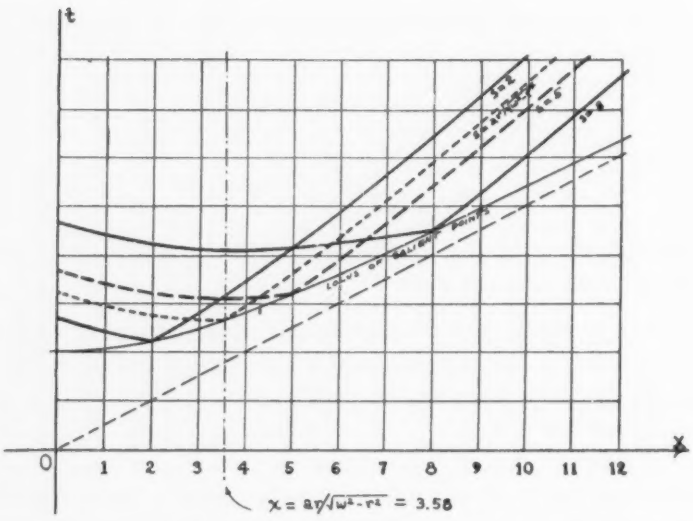
$$t=\frac{\sqrt{x^2+a^2}}{r}+\frac{\sqrt{(s-x)^2}}{w}$$

*We have thus far tacitly assumed that $r<w$, since $\sqrt{w^2-r^2}$ is imaginary and $\sin^{-1}(r/w)$ does not exist for $r>w$. However, the fact that t is a minimum for $x=s$ (for every positive s) when $r \geqq w$ is readily verified by noting that in such a case

$$t'(s+h)=\frac{s+h}{r\sqrt{(s+h)^2+a^2}}+\frac{1}{w}>0$$

and that

$$\begin{aligned} t'(s-h) &= \frac{s-h}{r\sqrt{(s-h)^2+a^2}}-\frac{1}{w} \\ &\leqq \frac{s-h}{r\sqrt{(s-h)^2+a^2}}-\frac{1}{r} \\ &= \frac{1}{r}\left[\frac{s-h-\sqrt{(s-h)^2+a^2}}{\sqrt{(s-h)^2+a^2}}\right] \\ &<0. \end{aligned}$$



by means of the s-discriminant relation, we obtain the equation of the locus of the salient points of the members of the family of curves $t=t(s,x)$, i. e.,

$$t=\sqrt{x^2+a^2}/r$$

which is the upper branch of the hyperbola

$$\frac{t^2}{a^2/r^2}-\frac{x^2}{a^2}=1.$$

The graph clearly illustrates how the minimum value of t is accompanied by a vanishing first derivative only when the condition

$$s>ar/\sqrt{w^2-r^2}$$

holds.

The Poristic Locus of the Gergonne Point

By Daniel Arany
Budapest, Hungary

In Article 34 of Gallaty's *Modern Geometry of the Triangle* the two statements are made: "The poristic locus of the Gergonne point M is a circle coaxal with $O(R)$ and $I(r)$", and "It may be shown that the radius of the M circle

$$= \frac{r(R-2r)}{4R+r},$$

but the proof is long." The object of this note is to give a short proof.

If we denote by m the center of the M circle, by E the point of intersection of OI with the radical axis of $O(R)$ and $I(r)$, the condition for the M circle to be coaxal with $O(R)$ and $I(r)$ is:

(1) $\overline{EL}^2 = \overline{Em}^2 - \rho^2$, where

(2) $$\overline{EL}^2 = \frac{r^3(4R+r)}{4d^2} = \frac{r^3(4R+r)^3}{4d^2(4R+r)^2}$$

(3) $$\overline{Em} = \overline{EO} - \overline{Om}$$

(4) $$\overline{EO} = \frac{2R^2-2Rr-r^2}{2d}$$

(5) $$\overline{Om} = \frac{4(R+r)d}{4R+r}$$

(6) $$\overline{Em} = \frac{2R^2r+10Rr^2-r^3}{2d(4R+r)}$$

(7) $$\overline{Em}^2 = \frac{4R^4r^2+40R^3r^3+96R^2r^4-20Rr^5+r^6}{4d^2(4R+r)^2}$$, and finally

(8) $$\overline{Em}^2 - \rho^2 = \frac{64R^3r^3+48R^2r^4+12Rr^5+r^6}{4d^2(4R+r)^2} = \overline{EL}^2 = \frac{r^3(4R+r)^3}{4d^2(4R+r)^2}$$

where ρ is the radius of the M circle. Formulas (2) and (4) appear in Article 23 of Gallaty's book.

Humanism and History of Mathematics

Edited by
G. WALDO DUNNINGTON

An Overlooked Numeral System of Antiquity*

By ALLEN A. SHAW
University of Arizona

1. It is the purpose of this paper to give a concise account of the Armenian numeral system which, to the best of my knowledge, has been overlooked by historians of mathematics. The Armenians have secured a permanent place in the history of civilization and, without doubt, their numeral system deserves a place in the history of mathematics for reasons to be seen presently.

2. Before the introduction of the "Hindu" numerals into Armenia in the 12th century, the Armenians realizing the inefficiency of the Persian, and the Greek (Herodianic) numeral systems, invented their own alphabetic numerals in the 4th century better adapted to the theory of numbers though more difficult for the trading class. This difficulty, however, was met by the use of the abacus.

3 The Armenian numeral system consisted in assigning the first nine letters of the alphabet to units, the second nine to tens, the third nine to hundreds, and the last nine to thousands as shown in the following table:

1	2	3	4	5	6	7	8	9	*Units*
ա	բ	գ	դ	ե	զ	է	ը	թ	
10	20	30	40	50	60	70	80	90	*Tens*
ժ	ի	լ	խ	ծ	կ	հ	ձ	ղ	
100	200	300	400	500	600	700	800	900	*Hundreds*
ճ	մ	յ	ն	շ	ո	չ	պ	ջ	
1000	2000	3000	4000	5000	6000	7000	8000	9000	*Thousands*
ռ	ս	վ	տ	ր	ց	ւ	փ	ք	

*See American Mathematical Society *Bulletin*, Vol. 37, p. 826.

4. Higher numbers were obtained by placing one of the above letters *to the left* of a greater symbol. e. g. ⵔ = 10,000, ⵚ = 100,000, ⵀ = 400,000, ⵏⵏ = 1,000,000, ⵎⵏⵏ = 200,000,000, etc. From the last two examples the reader may think that the Armenian repeated the symbols as the Romans did in cases like III, XX, CCC, MM, etc. But the two cases are different: the Romans repeated symbols for the purpose of *addition* only and this was done quite *generally*, for XX means 10+10, MM means 1000+1000, etc. While the Armenian repeated the letters *very rarely* and when they repeated it was for the purpose of *multiplication* only, for ⵏⵏ means 1000 × 1000, ⵎⵏⵏ = 200×1000×1000, etc. While ⵡⵒⵚⵟ = 4000+500+70+3 = 4573. This *juxtaposition* of *same* numerals in Armenian indicated multiplication and of unlike symbols indicated addition provided smaller units followed the larger ones. It is to be noted here that the Greeks also had the same rule in a few cases in the Herodianic system. e. g. $\overline{\Delta} = 5\times10 = 50$, $\overline{\text{H}} = 5\times100 = 500$, $\overline{\text{M}} = 5\times10{,}000 = 50{,}000$.

The Armenian alphabetic numerals were not difficult for purposes of calculation once the numerical value of each letter was known. In fact, for an Armenian it was considerably simpler to write 1010 by ⵑⵔ, 1100 by ⵑⵚ, 9090 by ⵠⵟ, etc. than by the "Hindu" system. The Armenians, like the Greeks, used both capital and small letters for numerals.

5. The Armenian numeral system had no zero. It had no place value in the sense we understand now for the "Indian" system. Still it is not correct to say that the system had no local value, for it *had* the place value in a limited sense as an examination of the above table and the illustrative examples will show. The larger symbols were placed *before* the smaller ones and this is a case of local value in a limited sense. We have here the germ of the very important notion that the value of a symbol may be periodic.

6. While the table in §3 is the one commonly used by the ancient and present Armenians, yet a careful examination of the works of Ananiah Shiragooni* indicates that there were *variant forms* of numbers very frequently used, involving *multiplicative* as well as additive principles which reduce the thirty-six symbols to twelve and thus place the Armenian notation on *decimal* basis without use of zero. The corresponding variant forms do *not* exist in the Semitic and Greek systems.

Now consider the following examples,† taken from Shiragooni's works, which reveal *multiplicative* and additive principles and show the frequent use of variant forms:

*or Ananiah of Shirag.

†These examples are taken from the MSS of Shiragooni's works in possession of the present writer.

ԻԺ = 20 (for Ի), ՐՃ = 200 (for Ր), ՔՃ = 300 (for Յ),
ՐՌ = 2000 (for Ս), ՊՌՍ = 4800 (for ՈՍ), ՁՌ = 6000 (for Ց),
ՂՌ = 8000 (for Փ), ՖՌ = 9000 (for Ք), ԺՐՌ = 12000 (for 12 "pure"),
ԺՂՌ = 18000, ԻՌՐ = 20200.

From the same author in the MS of St. Lazar, Venice, Italy, we have the following additional examples* which reveal multiplicative and additive principles in one and the same number:

ԻՌՁՃ = 20600, ԻՐՌԷՃ = 22700, ԻՂՌՂՃ = 28800, ԽՊՌՁՃ = 43600,
ՃՀՐՌՂՃ = 172800, ՐՃՊՌՊՃ = 204400, ՐՃԺՁՌ = 216000,
ՐՃՃՂՌ = 288000, ՔՃԽԽՌՁՃ = 345600, ՊՃԼՁՌ = 436000,
ՂՃԿԿՌ = 865000.

7. From the above examples we are able, by deduction, to construct the following table, which is *equivalent* to the table of §3 and has the advantage of giving higher numbers by the use of multiples† of Ռ:

1	2	3	4	5	6	7	8	9	*units*
Ա	Բ	Գ	Դ	Ե	Զ	Է	Ը	Թ	
10	20	30	40	50	60	70	80	90	*tens*
Ժ	ԲԺ	ԳԺ	ԴԺ	ԵԺ	ԶԺ	ԷԺ	ԸԺ	ԹԺ	
100	200	300	400	500	600	700	800	900	*hundreds*
Ճ	ԲՃ	ԳՃ	ԴՃ	ԵՃ	ԶՃ	ԷՃ	ԸՃ	ԹՃ	
1000	2000	3000	4000	5000	6000	7000	8000	9000	*thousands*
Ռ	ԲՌ	ԳՌ	ԴՌ	ԵՌ	ԶՌ	ԷՌ	ԸՌ	ԹՌ	
10000	20000	30000	40000	50000	60000	70000	80000	90000	*higher*
ՄՌ	ԻՌ	ԼՌ	ԽՌ	ԾՌ	ԿՌ	ՀՌ	ՃՌ	ՂՌ	*numbers*
100000		200000		300000		400000		500000	600000
ՃՌ		ՐՃՌ		ԳՃՌ		ԴՃՌ		ԵՃՌ	ԶՃՌ
or ՃՌ		ՍՌ		ՅՌ		ՆՌ		ՉՌ	ՈՈ

700000	800000	900000
ԷՃՌ	ԸՃՌ	ԹՃՌ
ՀՌ	ՊՌ	ՖՌ

Millions and larger numbers were usually written in words; e. g., for one million the ancient Armenians used hundred myriads (in Armenian: *haryure pure* = 100 × 10000), or sometimes in symbols: ՌՌ = 1000 × 1000. We also find ՐՈՌ for 200,000 = 200 myriads. The last two examples are not from Shiragooni's works, but from those

*Obtained through the courtesy of the Rev. P. Elia Pechigian of St. Lazar, Venice, Italy.

†Note that the Greeks used, instead, multiples of M, the myriad.

of the early fathers of St. Lazar, Venice, Italy, who must have seen them used, no doubt, by the ancient Armenians.

8. With regard to the actual use of these numerals in computations, the following examples will show that the processes were essentially the same as ours, and were quite easy to handle by use of the *variant* forms.

Example 1. To multiply մգ by դբ.

$$\text{մգ}\times\text{դբ}=(\text{մ}+\text{գ})(\text{դ}+\text{բ})$$
$$=\text{դ}(\text{դ}+\text{բ})+\text{գ}(\text{դ}+\text{բ})$$
$$=\text{բն}+\text{դմ}+\text{մդ}+\text{գլ}$$
$$=\text{բնբմլգ}$$

$$43\times52=(40+3)(50+2)$$
$$=40(50+2)+3(50+2)$$
$$=2000+80+150+6$$
$$=2236.$$

the last step, բնբմլգ, was probably obtained by the aid of the abacus.

Similarly the product of two or more numbers can be obtained by use of the Law of Distribution and by the abacus.

Example 2. To divide մգնմզ by լհ, by the factor method.

ե) մգն+մ+ն

հ) գն+բմ+ζ

նբմլգ

7) 43190

5) 6170

1234 = Quotient

or thus:

43000+100+90 (30+5

30000

5000

1000+200+30+4

8000+100+90

6000

1000

1000+100+90

900+100+50

100 +40

100 +20

+20

0

9. The study of the Armenian numerals has led the present writer to a *new theory* on the origin of our common numeral system. The reader now knows that the ancient Armenians could write any number N (using variant forms) as follows:

$$N=\cdots+a_4.10000+a_3.1000+a_2.100+a_1.10+a_0 \quad (a_i<10).$$

which in our system is written as

$$N = a_n \cdots a_4 a_3 a_2 a_1 a_0.$$

for example, the Armenians wrote 865423 as ՊՀԿՌՆԻԳ, almost as we do now.

The Armenian alphabetic system is more perfect than any of the other ancient alphabetic systems. The only missing symbol in the system is zero, which was discovered and used later in the 8th century. The present writer holds that the origin of our common numeral system is *Armenian*, or *Græco-Armenian*, worked out at the court of the Baghdad Califs, possibly under the patronage of Harun-al-Rashid (c. 786 A. D.). I hope to write on this subject a separate paper in the near future.

The following is an abstract of the paper *The Development of the Problem of Bolza in the Calculus of Variations* presented by Dr. Carroll P. Brady at the March, 1939 meeting of the Texas Section of the Mathematical Association of America:

Bliss formulated the problem of Bolza and discussed necessary conditions for the problem. Hestenes was the first to formulate a condition of Mayer by the use of which a sufficiency proof could be made for the problem of Bolza. Following Hestenes' sufficiency theorems Bliss, Reid, Morse, and Hestenes proposed other useful forms of the Mayer condition. The author, in his Chicago thesis, made use of the theory of the problem of Bolza. to study the problem of minimizing a function of integrals. The latter problem dates back to the time of Euler. As formulated by the author it is readily transformed into a problem of Bolza. However, an application of the sufficiency theorems for the latter problem yields only a restricted strong relative minimum for the former because the definitions of a strong relative minimum for the two problems are not equivalent. A satisfactory theorem concerning strong relative minima for the author's problem was deduced by the use of supplementary methods similar to those used by Hestenes for isoperimetric problems. The fact that the theory for the problem of Bolza failed to yield the results desired for the isoperimetric problem and for the author's problem led Hestenes last fall to a reformulation of the problem of Bolza.

The Teacher's Department

Edited by
JOSEPH SEIDLIN and JAMES MCGIFFERT

An Approach to Trigonometry

By A. L. O'TOOLE
Chicago, Illinois

In the February number* of this journal Professor Karpinski presented arguments in favor of giving the general definitions of the trigonometric functions at the very beginning of the course. This writer feels that the method recommended there is superior to that which begins with a chapter or two devoted exclusively to right triangles and for several years he has preferred not to use a text book which begins with the right triangle formulas.

But there is another point in the approach to college trigonometry which Professor Karpinski did not mention and which is of some interest. Although many text book writers and teachers begin with the general formulas for the functions, they may place too much emphasis on certain right triangles in setting up the definitions of the functions. Would it not be better to put the major emphasis on a

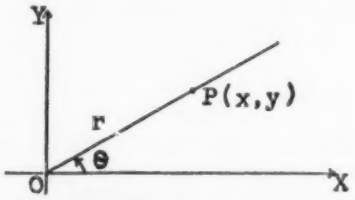


Figure I

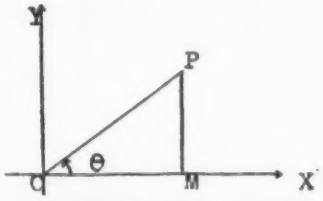


Figure II

point P on the terminal side of the angle, its coordinates x and y, and the distance r from the origin to P, as illustrated in Figure I, instead of on the right triangle OMP, as in Figure II? Marking M as prominently as P and drawing MP as heavily as the terminal side of the angle and writing such statements as $\tan \theta = MP/OM = y/x$ may not be de-

**A Problem of Presentation in Trigonometry*, by Louis C. Karpinski, NATIONAL MATHEMATICS MAGAZINE, Vol. XIII, No. 5, pp. 240-241.

sirable even at the beginning and may soon, as crutches often do, turn out to be more harmful than helpful. For example, emphasis on the right triangle *OMP*, when the angle is in the first quadrant, usually causes confusion for the student later when an angle terminating in the second quadrant is studied, for the angle then is not in the right triangle emphasized in that quadrant. If, on the other hand, the functions are thought of in connection with a figure like Figure I where the emphasis is on P, x, y and r, there should be less cause for confusion in dealing with angles terminating in quadrants other than the first and in developing the formulas for related angles.

It should not be inferred from what has just been written that the author would like to banish all right triangles from trigonometry. It is merely less emphasis that he suggests. One is using rectangular coordinates but the student should be able to distinguish between the number y and the straight line segment MP. The general formulas for the trigonometric functions are algebraic rather than geometric. The theorems on similar triangles and the Pythagorean theorem are necessary in this approach to trigonometry. One may take several positions of the point P on the terminal side of the angle and use similar right triangles to show that the ratios y/r, x/r and y/x are independent of the position of P on the terminal side. And one may use the Pythagorean theorem to prove that $x^2+y^2=r^2$. But after these notions have been made clear to the students what further necessity is there for the triangle *OMP* every time one wishes to discuss the functions of an angle? The fact that the triangle *OMP* of Figure II is not essential

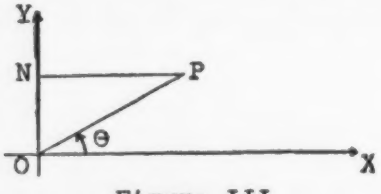


Figure III

is seen by noting that the right triangle *ONP* of Figure III would serve just as well. But, so far as the writer knows, no text book chooses to emphasize the triangle *ONP*. Isn't it possible that the great emphasis on the right triangle *OMP* instead of on the numbers x, y and r, and the point P is a carryover from the earlier text books which began with a non-graphical treatment of acute angles in right triangles?

As long as the present text books are in use it will be impossible to avoid the triangle *OMP* entirely, even if that were desirable. But, at least, the line *MP* in Figure II could be dotted or drawn very faintly in

comparison with *OX* and *OP* to indicate that it is only of secondary importance at the moment. Visual helps are to be encouraged only when they are not liable to become harmful crutches. The stress referred to here is illustrated by those text books which designate the triangle *OMP* with such a significant name as *the defining triangle*. Such emphasis may cause three difficulties. First, the angle in question may or may not be in the defining triangle. Second, the defining triangle disappears and the student is confused when he attempts to consider angles such as 0° and 90°. Third, it tends to detach the attention from the angle itself and focus it on a right triangle. There may be some psychological advantage in the learning process which encourages the student to keep his attention on the angle.

The writer has tried this approach a couple of times with college freshmen and it has introduced no great difficulties. The students seemed to like it.

The teaching of trigonometry by beginning with the general form of the definitions as suggested by Professor Karpinski and de-emphasizing the right triangle *OMP* as suggested here would seem to have the following desirable features. (1) It involves one set of formulas instead of two. (2) It requires one set of proofs for the fundamental identities instead of two. (3) There is less trouble with the angles of 0° and 90°. (4) There is less trouble with the functions of angles in quadrants other than the first. (5) The best meaning of an angle is used from the start and nothing has to be unlearned later. (6) It saves time by presenting one set of formulas and proofs instead of two and this saving may be important as, for instance, in the case of a school operating on the Quarter system. (7) It introduces the notions of functionality and generalization early and naturally. (8) It may be a motivating factor as it connects with the student's previous experience through graphs and thus takes advantage of the fact that, of all the mathematics studied before coming to college, graphing is the part which students in general like best and will study most enthusiastically. Demonstrative geometry, on the other hand, is the least well liked of all high school mathematics, and hence any approach to trigonometry which suggests demonstrative geometry is not likely to be popular with the majority of the students. (9) Working in the first quadrant is just as easy as working in a right triangle, and so the applied problems usually studied early in trigonometry are no more difficult under the plan discussed here. In fact the graphical setting of a problem is the one the student will use later in the course and in the more advanced courses in mathematics and science.

Mathematical World News

Edited by
L. J ADAMS

The W. P. A. of New York City has as one of its projects the computation of certain mathematical tables, such as a table of exponential functions to fifteen decimal places and a table of the first ten powers of the integers from 1 to 1000. Dr. Arnold N. Lowan is in charge. This is one part of a general project directed by Dr. Lyman Briggs, of the National Bureau of Standards.

The National Council of Teachers of Mathematics will meet in Berkeley and San Francisco, California, in conjunction with the summer meeting of the National Education Association, on July 5, 6.

The Southwestern Section and the Wisconsin Section of the Mathematical Association of America were scheduled to meet on May 6 at Alpine, Texas and Milwaukee, Wisconsin, respectively.

Professor Albert Einstein celebrated his sixtieth birthday on March 16, 1939. One of the events marking the occasion was a radio program from Oakland, California, on which Professor J. O. Oppenheimer was the speaker.

The Fifth International Congress for the Unity of Science is to be held at Harvard University on September 5-10, 1939. The central theme will be the role of logic in science. For more complete information concerning the meeting, please address Professor C. W. Morris, University of Chicago, Chicago, Illinois.

Professor Werner Heisenberg, University of Leipzig, will lecture on cosmic rays at Purdue University during the summer session from July 1 to July 22.

The National Education Association has initiated a National Science Committee which met in Cleveland, Ohio on February 23, 24. Mr. Ira C. Davis, University of Wisconsin, is general chairman. The next meeting will be held on May 12, 13 in Cleveland. The Central

Association of Science and Mathematics Teachers is represented by Mr. Harold H. Metcalf of Oak Park High School, Oak Park, Illinois.

The sixteenth annual meeting of the Indiana Section of the Mathematical Association of America was held at Ball State Teachers College in Muncie, Indiana on April 28-29, 1939. The papers presented were:

1. *Problem Making.* Professor C. K. Robbins, Purdue University.
2. *Order Relation in Fields.* Professor E. Artin, Indiana University.
3. *Some Technical Aspects of the Mathematics of Seismology.* Professor Cairns.
4. *On the Foundations of Mathematics.* David O. Schechter, Manchester College.
5. *The Generalization of the Eckardt Point.* Dr. D. R. Shreve, Purdue University.
6. *The General Second Degree Equation Without Transformation of Coordinates.* Professor K. P. Williams, Indiana University.
7. *A Certain Lagrange Interpolation Polynomial.* Dr. Merritt S. Webster, Purdue University.

Fascicule XCV of the *Memorial des Sciences Mathématiques* is devoted to a paper by M. Ervin Feldheim on *Theorie de la convergence des procédés d'interpolation et de quadrature mécanique.*

Dr. Ferdinand von Lindemann died in Munich on March 7, 1939. He was the first to prove the impossibility of squaring the circle, that is, he proved the transcendence of pi. As a result of this he was awarded a prize by the Prussian Academy of Sciences.

A history of Hindu mathematics has been published by Bibhutibhusan Datta and Avadhesh Narayan Singh. The first volume was published in 1935, and the second is now available. The book may be obtained through Lahore: Motilal Banarsi Das. A third volume is contemplated. Since the work is essentially a source book, it will be welcomed by mathematical historians.

Professor G. R. Livingston, San Diego State College, was selected as chairman of the Southern California section of the M. A. A. for the coming year.

The annual meeting of the Mathematical Association of England was held at King's College, London on January 2-3, 1939. Mr. W. C. Fletcher was elected president for the year 1939. The retiring president, Mr. Hope-Jones, delivered an address on *Simplicity and Truthfulness in Arithmetic*. Other addresses included the following:

1. *The Teaching of Applied Mathematics in Technical Colleges.* Mr. A. Buxton.
2. *The Symmetry of Patterns.* Professor W. L. Bragg.
3. *The Second Report on the Teaching of Geometry.* Miss M. A. Hooke.

The Academy of Sciences met in Paris, France on January 30, 1939. Among the mathematical papers presented were:

1. *The variety of plane generators of a real quadric and the topology of an orthogonal group of n variables.* C. Ehresmann.
2. *A differential equation for which the point* $x=y=0$ *is a focus.* L. Dehousse.

The American Mathematical Society met at Stanford University, California on April 15, 1939.

Instructors wishing to investigate exchange positions with teachers in England should address: Honorary Secretary, British Joint Committee for Interchange, 37 Charles Street, Berkeley Square, London, W. I.

The Royal Society of Edinburgh met on February 6, 1939. One of the addresses was *The relation between mathematics and physics* by P. A. M. Dirac.

The ordinary membership in the American Mathematical Society is now 2,000. During 1938 the total attendance at all meetings was 1,435. The number of papers read was 475. The officers for the year 1939 include: *President*, Professor G. C. Evans; *Vice-President*, Professor C. R. Adams; *Associate Secretary*, Professor T. R. Hollcroft; *Treasurer*, Professor B. P. Gill; *Librarian*, Professor R. C. Archibald.

Problem Department

Edited by
ROBERT C. YATES and EMORY P. STARKE

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscript be typewritten with double spacing. Send all communications to ROBERT C. YATES, College Park, Md.

SOLUTIONS

No. 187. Proposed by *V. Thébault*, Le Mans, France.

Let A_1, B_1, C_1 be the middle-points of the sides BC, CA, AB of a triangle ABC inscribed to a circle with center O. Prolong the lines OA_1, OB_1, OC_1, to lengths $AA'=\frac{1}{2}r_a$, $BB'=\frac{1}{2}r_b$, $CC'=\frac{1}{2}r_c$ where r_a, r_b, r_c are the radii of the circles escribed to ABC

(1) Show that the circumcircles of $A'BC$, $B'CA$, $C'AB$ are tangent to the inscribed circle of ABC and to the circumcircle of $A'B'C'$.

(2) Examine the case where the radius of one of the escribed circles is replaced by that of the inscribed circle of the fundamental triangle.

Solution by *C. A. Balof*, Lincoln College, Illinois.

There seems to be an error in the statement of this problem. Apparently AA', BB', CC' should be replaced by A_1A', B_1B', C_1C', respectively. With these changes, we have the following solution:

(1) $$OA_1=R\cos A,\ OB_1=R\cos B,\ OC_1=R\cos C,$$

$$r_a=s\tan\tfrac{1}{2}A=R(1-\cos A+\cos B+\cos C)$$

$$r_b=s\tan\tfrac{1}{2}B=R(1+\cos A-\cos B+\cos C)$$

$$r_c=s\tan\tfrac{1}{2}C=R(1+\cos A+\cos B-\cos C),$$

where R is the circumradius and s is one-half the perimeter of ABC. Hence $OA'=OB'=OC'=R(1+\cos A+\cos B+\cos C)/2$ and O is the circumcenter of $A'B'C'$.*

*It is not difficult to show that the radius of the circumcircle of $A'B'C'$ is $(R+r/2)$.—ED.

Now O', the circumcenter of $A'BC$, lies on OA'. Hence the normal to OA' at A' is a common tangent to the circumcircles of $A'B'C'$ and $A'BC$, and therefore these circles are tangent at this point. Similarly, the circumcircles of $B'CA$ and $C'AB$ are tangent to the circumcircle of $A'B'C'$ at B' and C' respectively.

Let I be the incenter, r the inradius, of ABC. Then

$$O'A' = O'C = (a^2+r_a^2)/4r_a, \quad O'A_1 = \pm(a^2-r_a^2)/4r_a.$$

Applying the Law of Cosines to triangle ICO', angle ICO' being either the sum or difference of angles $\frac{1}{2}C$ and $O'CB$, we have

$$(O'I)^2 = [a^4+2a^2r_a^2+r_a^4-8a^2\cdot r\cdot r_a-8r\,r_a^3 + 16r^2r_a^2(\csc^2\tfrac{1}{2}C - a\cot\tfrac{1}{2}C/r+r_a/r)]/16r_a^2.$$

But

$$\csc^2\tfrac{1}{2}C - a\cot\tfrac{1}{2}C/r+r_a/r = ab/(s-a)(s-b)-a(s-c)/r^2+s/(s-a) = 1.$$

Hence $(O'I)^2 = (a^4+2a^2r_a^2+r_a^4-8r\,r_a^3-8a^2r\,r_a+16r^2\,r_a^2)/16r_a^2$.

Then $O'I = (a^2+r_a^2)/4r_a - r = O'A' - r$. The distance between their centers being the difference of their radii, the circumcircle of $A'BC$ is tangent to the incircle of ABC. Similarly, the circumcircles of $B'CA$ and $C'AB$ are tangent to the incircle of ABC.

(2) Let r_a be replaced by r. If the circumcircles of $A'B'C'$ and $A'BC$ are tangent, the point of tangency is A'. Then the circumcenter of $A'B'C'$ is on OA', at O the point of intersection of OA' and the perpendicular bisector of $B'C'$. But O is not the circumcenter of $A'B'C'$, since $r_a \neq r$. Therefore, in this case, the circumcircles of $A'B'C'$ and $A'BC$ are not tangent.

As in (1), we have $O'A' = O'C = (a^2+r^2)/4r$, $O'A_1 = (a^2-r^2)/4r$. $(O'I)^2 = (a^4+9r^4-6a^2r^2+16r^3(r\csc^2\tfrac{1}{2}C - a\cot\tfrac{1}{2}C))/16r^2$.

But $$(O'A'-r)^2 = a^4+9r^4-6a^2r^2)/16r^2.$$

Hence, since $r\csc^2\tfrac{1}{2}C - a\cot\tfrac{1}{2}C \neq 0$, $O'I \neq O'A' - r$ and the circumcircle of $A'BC$ is not tangent to the incircle of ABC.

Also solved by the *Proposer*.

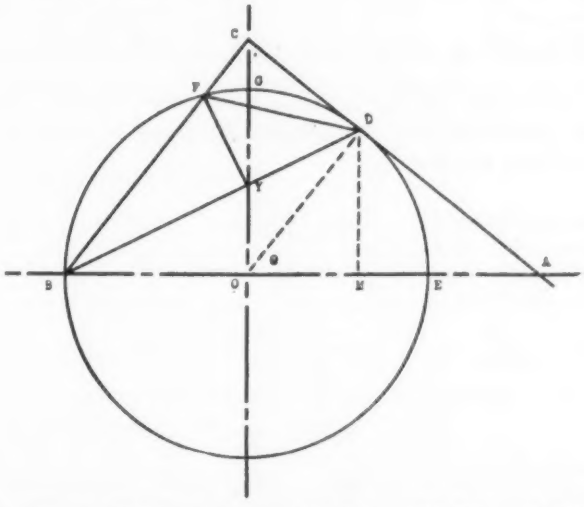
No. 247. Proposed by *Walter B. Clarke*, San Jose, California.

Consider only triangles having circumcenter lying on a side of the orthic triangle.

(1) What is the largest possible angle in such a triangle?

(2) Construct one that is isosceles.

Discussion* by the *Proposer*.



Part I: Draw the two perpendicular radii *BO* and *GO* as shown. Construct the tangent *CA* to the circle at *D* perpendicular to *BC*, with *C* on *OG*. Let *BC* intersect the circle at *F* and the two lines *BD* and *OG* intersect at *Y*. It is not difficult to show that *FY* is perpendicular to *BD*.

Consider triangle *BFD*. Clearly *C* is one vertex of its orthic triangle, and *Y* another vertex. Thus *CY* is a side of the orthic triangle and contains the center *O* of the circumcircle of *BFD*. This construction offers the triangle of the type considered which has maximum angle: $\measuredangle BFD = \pi/2 + (1/2) \arccos(\sqrt{5}-1)/2 = 115°55'$, approximately.

The triangle *ABC* has many unexpected features, not the least interesting of which is that *EA* is the side of a regular decagon inscribed in the circle.

Editor's Note: If the circle has unit radius, the requirement that *AC* and *BC* be perpendicular is that

$$(OC)^2 = (BO)(OA) = OA = OM + MA,$$

or $\tan^2\theta = \sec\theta$, where $\theta = \measuredangle DOM$.

That is, $\cos^2\theta + \cos\theta - 1 = 0$,

from which $\cos\theta = (\sqrt{5}-1)/2 = 2\sin 18°$.

*Cf. solution by *C. W. Trigg*, this Magazine, Vol. XIII, No. 5, February, 1939, p. 249.

Construction by straightedge and compasses is thus possible by first erecting a segment of length 2 tangent to the circle at E. It is easily verified that $\cos\theta = OM = EA$.

No. 257. Proposed by *W. V. Parker*, Louisiana State University.

(1) If $A_1(x_1,y_1)$, $A_2(x_2,y_2)$, $A_3(x_3,y_3)$ are the vertices of a triangle and a_i is the length of the side opposite A_i, show that the equation of the circumscribing circle is:

$$a_1^2\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} \cdot \begin{vmatrix} x & y & 1 \\ x_3 & y_3 & 1 \\ x_1 & y_1 & 1 \end{vmatrix} + a_2^2\begin{vmatrix} x & y & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \cdot \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} + a_3^2\begin{vmatrix} x & y & 1 \\ x_3 & y_3 & 1 \\ x_1 & y_1 & 1 \end{vmatrix} \cdot \begin{vmatrix} x & y & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

(2) Prove that there is one, and only one, ellipse circumscribing the triangle $A_1A_2A_3$ for which this triangle is an inscribed triangle of maximum area. Prove that the equation of this ellipse is:

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} \cdot \begin{vmatrix} x & y & 1 \\ x_3 & y_3 & 1 \\ x_1 & y_1 & 1 \end{vmatrix} + \begin{vmatrix} x & y & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \cdot \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} + \begin{vmatrix} x & y & 1 \\ x_3 & y_3 & 1 \\ x_1 & y_1 & 1 \end{vmatrix} \cdot \begin{vmatrix} x & y & 1 \\ x_2 & x_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

(3) Show that an hyperbola having its center at the centroid of a triangle cannot pass through all three vertices of the triangle.

Solution by the *Proposer*.

Write $$L_1 = \begin{vmatrix} x & y & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}, \quad L_2 = \begin{vmatrix} x & y & 1 \\ x_3 & y_3 & 1 \\ x_1 & y_1 & 1 \end{vmatrix}, \quad L_3 = \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}$$

and let p_i, q_i, r_i be defined by $L_i \equiv p_i x + q_i y + r_i$. Let $s = r_1 + r_2 + r_3$ and we have $a_i^2 = p_i^2 + q_i^2$, $p_1 + p_2 + p_3 = q_1 + q_2 + q_3 = 0$,

$$\begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix} = \begin{vmatrix} p_2 & p_3 \\ q_2 & q_3 \end{vmatrix} = \begin{vmatrix} p_3 & p_1 \\ q_3 & q_1 \end{vmatrix} = s \neq 0.$$

The equation of the family of conics through A_1, A_2, A_3 is

$$F \equiv K_1 L_2 L_3 + K_2 L_3 L_1 + K_3 L_1 L_2 = 0,$$

where K_1, K_2, K_3 are parameters.

The conic is a circle if the coefficient of xy is zero and the coefficient of x^2 is equal to the coefficient of y^2. That is, if

$$(p_2 q_3 + p_3 q_2) K_1 + (p_3 q_1 + p_1 q_3) K_2 + (p_1 q_2 + p_2 q_1) K_3 = 0,$$

$$(p_2 p_3 - q_2 q_3) K_1 + (p_3 p_1 - q_3 q_1) K_2 + (p_1 p_2 - q_1 q_2) K_3 = 0.$$

We note that the successive terms in each of these equations are formed by cyclic permutation of the subscripts. Since

$$\begin{vmatrix} p_3 q_1 + p_1 q_3 & p_1 q_2 + p_2 q_1 \\ p_3 p_1 - q_3 q_1 & p_1 p_2 - q_1 q_2 \end{vmatrix} = (p_1^2 + q_1^2) \begin{vmatrix} p_2 & p_3 \\ q_2 & q_3 \end{vmatrix} = a_1^2 s,$$

it follows that $K_1 : K_2 : K_3 = a_1^2 : a_2^2 : a_3^2$ and we have the first required result.

A triangle inscribed in an ellipse is an inscribed triangle of maximum area if its centroid is at the center of the ellipse. The center of any conic of the above family is the intersection of the two lines

$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0.$$

If we substitute the coordinates of the centroid of $A_1 A_2 A_3$ in the expressions above we get $L_1 = L_2 = L_3 = s/3$. The values of K_1, K_2, K_3 for the conic with center at the centroid of $A_1 A_2 A_3$ are given by the equations

$$(p_2 + p_3) K_1 + (p_3 + p_1) K_2 + (p_1 + p_2) K_3 = 0.$$

$$(q_2 + q_3) K_1 + (q_3 + q_1) K_2 + (q_1 + q_2) K_3 = 0.$$

Again the successive terms of these equations are formed by a cyclic permutation of subscripts. Since

$$\begin{vmatrix} p_3 + p_1 & p_1 + p_2 \\ q_3 + q_1 & q_1 + q_2 \end{vmatrix} = s,$$

it follows that $K_1 = K_2 = K_3$ in this case and we have the second required result.

We have seen that there is one, and only one, conic of the family through A_1, A_2, A_3 with center at the centroid of the triangle. Its equation is $L_2L_3+L_3L_1+L_1L_2=0$ and it is always an ellipse since

$$(p_2q_3+p_3q_2+p_3q_1+p_1q_3+p_1q_2+p_2q_1)^2$$
$$-4(p_1p_2+p_2p_3+p_3p_1)(q_1q_2+q_2q_3+q_3q_1)$$
$$=(p_1q_1+p_2q_2+p_3q_3)^2-(p_1^2+p_2^2+p_3^2)(q_1^2+q_2^2+q_3^2)$$
$$=-\begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix}-\begin{vmatrix} p_2 & p_3 \\ q_2 & q_3 \end{vmatrix}-\begin{vmatrix} p_3 & p_1 \\ q_3 & q_1 \end{vmatrix}=-3s^2.$$

This follows at once by squaring each of $p_1+p_2+p_3=0$, $q_1+q_2+q_3=0$, and forming their products. Therefore, there is no hyperbola with center at the centroid of the triangle and passing through the three vertices.

Also solved by *G. F. Alrich* and *Johannes Mahrenholz.*

No. 265. Proposed by *J. Rosenbaum*, Bloomfield, Connecticut.

Prove (or disprove): The sum of the pth powers of the distances to the sides of a regular n-gon from a point on its incircle is constant with regard to the position of the point, provided p is a positive integer less than n. This constant is given by

$$\sum_{i=1}^{n} |d_i|^p=\frac{1\cdot3\cdot5\cdot\ldots\cdot(2p-1)}{p!}\cdot n\cdot r^p,$$

where r is the inradius of the n-gon.

Solution by the *Editors.*

With the origin at the center and the $X-$axis perpendicular to one side of the n-gon, the equations of the sides are

$$x\cos(2k\pi/n)+y\sin(2k\pi/n)-r=0, \qquad k=0, 1, \ldots, n-1.$$

By the familiar method of coordinate geometry, we find that the distances from any point $P:(r\cos\theta, r\sin\theta)$ on the circle to the sides are

$$|d_k|=|r\cos\theta\cos(2\pi k/n)+r\sin\theta\sin(2\pi k/n)-r|=2r\sin^2(\pi k/n-\theta/2).$$

Thus the desired sum is

$$\sum_{k=0}^{n-1}|d_k|^p=(2r)^p\sum_{k=0}^{n-1}\sin^{2p}(\pi k/n-\theta/2). \tag{1}$$

Further simplification is effected by two formulas from trigonometry*

$$(2)\quad \sin^{2p}A=(-1)^p 2^{1-2p}\sum_{j=0}^{p-1}(-1)^j\binom{2p}{j}\cos(2p-2j)A+2^{-2p}\binom{2p}{p},$$

$$(3)\quad \sum_{k=0}^{n-1}\cos(\alpha+k\beta)=\cos\left(\alpha+\frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}\operatorname{cosec}\frac{\beta}{2}.$$

If (2) be employed in (1), there results

$$(4)\quad \sum_{k=0}^{n-1}|d_k|^p=(-1)^p(2r)^p 2^{1-2p}\sum_{j=0}^{p-1}(-1)^j\binom{2p}{j}\sum_{k=0}^{n-1}\cos\left\{(2p-2j)\left(\frac{\pi k}{n}-\frac{\theta}{2}\right)\right\}+2^{-p}nr^p\binom{2p}{p}.$$

If $p<n$, we have

$$(5)\quad \sum_{k=0}^{n-1}\cos\left\{(2p-2j)\left(\frac{\pi k}{n}-\frac{\theta}{2}\right)\right\}=\sum_{k=0}^{n-1}\cos\{(j-p)\theta+(p-j)2\pi k/n\}=0$$

because of the factor $\sin(n\beta/2)\operatorname{cosec}(\beta/2)$ in (3).

Using (5) and (4), we have the proposed formula in the equivalent form

$$\sum_{k=0}^{n-1}|d_k|^p=2^{-p}nr^p\binom{2p}{p}.$$

Note (A). If $p\geqq n$ there exist one or more values of j such that $p-j$ is a multiple of n; say, $p-j=cn$, $c=1,2,\ldots,[p/n]$. For all other values of j, (5) applies. For these exceptional values of j (3) cannot be used because of the factor $\operatorname{cosec}\beta/2=\operatorname{cosec}2\pi kc$. But for such values of j the left member of (3) can be found directly and is $n\cos cn\theta$. Thus $\sum|d_k|^p$ varies with the choice of point P.

Note (B). In the "additional discussion" under problem 3774, *American Mathematical Monthly*, March, 1938, referred to by the Proposer, the converse of our present problem and the method of proof are suggested. The restriction, there stated, that p must be even applies only to the more general discussion and is unnecessary in the case of the incircle.

*See, for example, Hobson, *Plane Trigonometry*, pp. 54, 90.

No. 266. Proposed by *J. Rosenbaum*, Bloomfield, Connecticut.

Find all positive integral values of n which satisfy the equation:

$$1^n+2^n+3^n+2^{2n-1}+(2+\sqrt{3})^n+(2-\sqrt{3})^n=\frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!}\cdot 2^{n+1}3.$$

Solution by the *Editors.*

In equations (1) and (4) of the above solution, put 2, n, 12, 0 for r, p, n, Θ, respectively. Using $\sin \pi/12=\frac{1}{2}(2-\sqrt{3})^{1/2}=\sin 11\pi/12$ and $\sin 5\pi/12=\frac{1}{2}(2+3)^{1/2}=\sin 7\pi/12$, and then dividing by 2, we have

$$\text{(I)}\quad (2-\sqrt{3})^n+1^n+2^n+3^n+(2+\sqrt{3})^n+2^{2n-1}=2^{2n-1}\sum_{k=0}^{11}\sin^{2n}(\pi k/12)$$
$$=(-1)^n\sum_{j=0}^{n-1}(-1)^j\binom{2n}{j}\sum_{k=0}^{11}\cos(n-j)\pi k/6+6\binom{2n}{n}.$$

As in the discussion of the above problem, for all $n<12$ we have

$$\text{(II)}\qquad \sum_{k=0}^{11}\cos(n-j)\pi k/6=0$$

so that the proposed formula is true. But for every $n>11$, there exist values of j for which we may put $n-j=12c$, $c=1,2,\ldots,[n/12]$. For such values of j we have

$$\text{(III)}\qquad \sum_{k=0}^{11}\cos(n-j)\pi k/6=\sum_{k=0}^{11}\cos 2ck\pi=12.$$

For other values of j, (II) still holds. Since also $n-j$ is even whenever (II) is replaced by (III), the last member of (I) is composed of certain positive multiples of 12 in addition to the term $6\binom{2n}{n}$. Hence for all $n>11$, the left member of (I) is greater than $6\binom{2n}{n}$.

No. 267. Proposed by *Arnold Emch*, University of Illinois.

Theorem: Given an equilateral hyperbola, H, with vertices A and B. Join any point P on H to A and B. Prove that the bisecting line of angle APB is parallel to one of the asymptotes.

I. Solution by *H. T. R. Aude*, Colgate University.

This theorem can be extended by letting the points A and B be any two diametrically opposite points of the hyperbola.

Let the equation $xy=c$ represent the curve. For the points P, A, and B choose the coordinates (x,y), (x_1,y_1), and $(-x_1,-y_1)$. Then the slopes of the lines AP and BP are respectively:

$$(y-y_1)/(x-x_1)=(c/x-c/x_1)/(x-x_1)=-c/xx_1;$$

$$(y+y_1)/(x+x_1)=c/xx_1.$$

It follows, since these inclinations are supplementary, that the bisecting line of angle APB is parallel to an asymptote.

A corollary presents itself: If the lines AP and BP are extended until they cut both asymptotes, then the four line segments between the asymptotes and the hyperbola are equal.

II. Solution by *D. L. MacKay*, Evander Childs High School, New York.

The theorem is true for any diameter AB. Let PB cut the asymptotes a and a_1 in D and D_1 and PB cut them in F and F_1, respectively, and draw AK and PL perpendicular to a and AK_1 and PL_1 perpendicular to a_1.

Since triangles PLF and AKF, PL_1F_1 and AK_1F_1 are similar, we have $PL/AK=PF/AF$, $PL_1/AK_1=PF_1/AF_1$. Multiplying term by term and noting that $(PL)(PL_1)=(AK)(AK_1)$, it follows that $(PF)(PF_1)=(AF)(AF_1)$ or $PF/AF=AF_1/PF_1$. Hence

$$AP/AF=AP/PF_1 \text{ and } AF=PF_1.$$

Draw BG parallel to AP to cut a in G. If O is the center of the hyperbola, then triangles BGO and AFO are congruent, $BG=AF=PF_1$, $GO=OF$ and PB is parallel to F_1G. Hence $FF_1=GF_1$ and since a_1 is the bisector of angle FF_1G it follows that the internal bisector of angle APB is parallel to a_1 and the external bisector is parallel to a.

III. Solution by *Eunice Lewis*, University of Oklahoma.

The asymptotes of an hyperbola are the double elements of the involution of conjugate diameters; hence the asymptotes separate harmonically every pair of conjugate diameters.* In particular, in an equilateral hyperbola, the asymptotes are the bisectors of the angles formed by every pair of conjugate diameters.* Now the lines joining any point on an hyperbola to the extremities of any diamteter are parallel to a pair of conjugate diameters.* Hence, if the hyperbola is equilateral, the bisectors of the angles formed by these lines are parallel to the asymptotes.

*Boyd Crumrine Patterson, *Projective Geometry*, John Wiley and Sons, Inc., 1937, Article 10.6, Theorem 1; Article 10.3; Article 10.5, Theorem 4.

Also solved by *Annie M. H. Christensen, Albert Farnell, A. E. Gault, Dorothy Graham, L. M. Kelly, Annie L. Lang, Johannes Mahrenholz, Lawrence E. Malvern, Fred Marer, H. P. Pettit, Cleon C. Richtmeyer, W. Irwin Thompson,* and *C. W. Trigg.*

The following references were furnished by the readers whose names appear in parentheses:

Besant: *Conic Sections Treated Geometrically,* 2nd Ed., Cambridge, 1869, Prob. 6, p. 138. (*T. A. A. Broadbent*).

Casey: *Treatise on the Analytical Geometry of Poini, Line, Circle and Conic Sections,* Prob. 9, p. 179. (*J. R. Musselman*).

Lemaire, J. *L'Hyperbole Équilatère,* 1927, p. 11. (*D. L. MacKay*).

Loney, S. L.: *Coordinate Geometry,* London, Macmillan, 1906, Illustrative Prob., p. 200, Art. 231. (*C. S. Carlson*).

Poncelet knew of the property that the angle subtended by an arc of an equilateral hyperbola is the same at any two diametrically opposite points of the curve. The theorem of the problem comes directly from this. (*T. Dantzig*).

Smith, C.: *An Elementary Treatise on Conic Sections,* London, Macmillan, 1884, Prob. 13, p. 163. (*C. W. Trigg*).

Fine and Thompson: *Coordinate Geometry,* 1909, Prob. 39, p. 117. (*A. E. Gault*).

No. 268. Proposed by *H. T. R. Aude,* Colgate University.

For the equation $x^2+xy+y^2=k$, only the positive integer solutions with $x<y$ are to be considered. The problem is to find the least values of k such that the equation has, respectively, two, three, four, solutions. Also find these solutions.

Solution by the *Proposer.*

Place $f(x,y)=x^2+xy+y^2$; then the following relations are easily verified:

(1) $$f(a,b)\cdot f(c,d)=f(bd-ac,\ ac+bc+ad) \text{ and}$$
$$f(a,b)\cdot f(c,d)=f(bc-ad,\ ad+bd+ac) \text{ or}$$
$$f(ad-bc,\ bc+bd+ac)$$

according as $bc-ad$ is positive or negative;

(2) $$\{f(a,b)\}^2=f(b^2-a^2,\ a^2+2ab).$$

The three smallest values of $f(x,y)$ are $f(1,2)=7$, $f(1,3)=13$, $f(2,3)=19$. Multiplying together the two smallest values of $f(x,y)$ gives, by use of (1)

$$f(1,2)\cdot f(1,3)=91=f(1,9)=f(5,6).$$

It follows that $k=91$ is the least value such that the proposed equation has two solutions.

Similarly we find

$$f(1,2)\cdot f(1,2)\cdot f(1,3)=637=f(4,23)=f(7,21)=f(12,17)$$

$$f(1,2)\cdot f(1,3)\cdot f(2,3)=1729=f(23,25)=f(15,32)=f(8,37)=f(3,40).$$

Thus $k=637$ and $k=1729$ are the least values for which there are three and four solutions respectively.

Also solved by *Johannes Mahrenholz* and *G. W. Wishard.*

Editor's Note: The above solution is based on the assumption that the representations of a composite number N in the form x^2+xy+y^2 obtained by applying equations (1) and (2) to the prime factors of N are the only possible representations. This is in fact true and is a consequence of the following lemmas. If p is a prime of the form x^2+xy+y^2, $0<x<y$, and $N=pm$ is of the same form, then m is either of the same form or is a perfect square. Any prime number not of the form $3k-1$ has a unique representation in the form x^2+xy+y^2. If N has no factor of the form $3k-1$ and if $N=N_1\cdot N_2$, then every representation of N in the form x^2+xy+y^2 is obtained by use of (1) and (2) from representations of N_1 and N_2 in the same form. See Dickson, *Elements of the Theory of Numbers*, p. 77, exercises 3, 4, and the discussion of the preceding pages. See also the analogous discussion for numbers of the form x^2+y^2 given in great detail in Carmichael, *Diophantine Analysis*, pp. 24-25 and 39-43.

No. 269. Proposed by *V. Thébault*, Le Mans, France.

Form a perfect square of four digits such that the sum of the two periods of two digits each will equal the number formed by reversing the digits of the root of the square.

Solution by *C. W. Trigg*, Los Angeles City College.

Put $N^2=(mn)^2=abcd$; then $ab+cd=nm$. It follows that

$$(mn)^2\equiv(m+n)^2\equiv a+b+c+d \pmod 9$$

and

$$nm\equiv m+n\equiv a+b+c+d \pmod 9.$$

Hence $(m+n)^2\equiv(m+n)$ or $m+n\equiv 0, 1 \pmod 9$.

Now also $1024\leqq N^2\leqq 9801$, so that $32\leqq N\leqq 99$. There are only fifteen values of $N=mn$ which satisfy these last two relations, and when these are examined further, the three solutions are found, viz.

$$(54)^2=2916,\ (55)^2=3021 \text{ and } (99)^2=9801.$$

Also solved by *Albert Farnell, Johannes Mahrenholz, G. W. Wishard,* and the *Proposer.*

No. 270. Proposed by *O. E. Lancaster,* University of Maryland.

Show that

$$n^{n-1}-(n-1)^n+(n-1)(n-2)^n/2!-(n-1)(n-2)(n-3)^n/3!+\ldots \pm(n-1)2^{n-1}\mp 1=(n-1)!$$

I: Solution by *Johannes Mahrenholz,* Gottbus, Germany.

Take $u=x(x-1)^{n-1}$, $u_1=xdu/dx$, $u_2=xdu_1/dx,\ldots,u_{n-1}=xdu_{n-2}/dx$.

Then $u=\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}x^{n-j}$, $u_1=\sum_{j=0}^{n-1}(-1)^j(n-j)\binom{n-1}{j}x^{n-j}$,

$u_2=\sum_{j=0}^{n-1}(-1)^j(n-j)^2\binom{n-1}{j}x^{n-j},\ldots,\ u_{n-1}=\sum_{j=0}^{n-1}(-1)^j(n-j)^{n-1}\binom{n-1}{j}x^{n-j}$.

The left member of the proposed equation is seen to be the value of u_{n-1} for $x=1$.

On the other hand, $u_1=xdu/dx=x(x-1)^{n-1}+(n-1)x^2(x-1)^{n-2}$,

$u_2=x(x-1)^{n-1}+3(n-1)x^2(x-1)^{n-2}+(n-1)(n-2)x^3(x-1)^{n-3}$,

$u_3=x(x-1)^{n-1}+7(n-1)x^2(x-1)^{n-2}+6(n-1)(n-2)x^3(x-1)^{n-3}$
$+(n-1)(n-2)(n-3)x^4(x-1)^{n-4}$,

. . . . . .

$u_{n-1}=x(x-1)^{n-1}+a_2(n-1)x^2(x-1)^{n-2}+a_3(n-1)(n-2)x^3(x-1)^{n-3}+$
$\ldots+a_{n-1}(n-1)(n-2)\cdots3\cdot2x^{n-1}(x-1)+(n-1)(n-2)\cdots3\cdot2\cdot1x^n$.

Thus, regardless of the numerical values of $a_2, a_3,\ldots, a_{n-1}$, the value of u_{n-1} for $x=1$ is $(n-1)!$ These two results complete the proof.

II. Solution by *J. Rosenbaum,* Bloomfield, Connecticut.

Denote the expression $\sum_{j=0}^{n}(-1)^j\binom{n}{j}(a-j)^p$ by $[n,p,a]$. It is observed that the left member of the equation in the problem is

$$[n-1,\ n-1,\ n].$$

Let there exist one positive integer, k, such that

(1) $\qquad [k,\ p,\ a]=0$, for $0\leqq p<k$ and for all a,

and also

(2) $\qquad [k,\ k,\ a]=k!$, for all a.

On account of the absence of restrictions on a, (1) can be written as

$$[k, p, a-1]=0. \tag{3}$$

The subtraction of (3) from (1) yields

$$[k+1, p, a]=0,\ 0\leqq p<k. \tag{4}$$

In the same manner as (1) was shown to imply (4), equation (2) can be shown to imply

$$[k+1, k, a]=0. \tag{5}$$

Equations (4) and (5) can be combined as

$$[k+1, p, a]=0,\ 0\leqq p<k+1. \tag{6}$$

Next observe that

$$(k+1)[k, k, k+1]=[k+1, k+1, k+1]. \tag{7}$$

By use of (2), equation (7) becomes

$$[k+1, k+1, k+1]=(k+1)!$$

Finally, by collecting like powers of a it is found that

$$[k+1, k+1, a]=(-1)^{k+1}\sum_{j=0}^{k+1}(-1)^j\binom{k+1}{j}a^{k-j+1}[k+1, j, k+1]. \tag{9}$$

By use of (6) and (8), (9) is reduced to

$$[k+1, k+1, a]=(k+1)!\text{ for all } a. \tag{10}$$

The proof that (1) and (2) hold for all values of k is completed in equations (6) and (10) together with the fact that (1) and (2) hold for $k=1$.

It is not difficult to modify the above proof to establish the more general equations

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}(a-jd)^p=0,\quad 0\leqq p<n,$$

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}(a-jd)^n=d^n\cdot n!$$

This implies further: If $f(x)$ is a polynomial of degree n, with unity for the coefficient of the highest power, then

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}f(x-jd)=d^n\cdot n!$$

Also solved by *Fred Fender* and *M. E. Wescott*. Solutions of the problem may also be found in Boole, *Finite Differences*, pp. 19-20, and in Davidson, *Higher Algebra*, p. 287.

No. 271. Proposed by *Robert C. Yates*, University of Maryland.

Given two distinct points A and B. Construct their midpoint by means of compasses alone.

I. Solution by *Johannes Mahrenholz*, Gottbus, Germany.

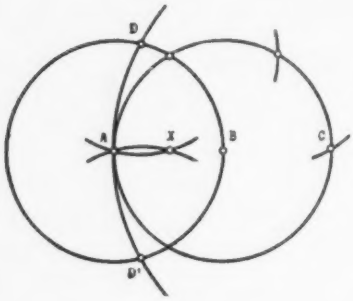


Fig. I.

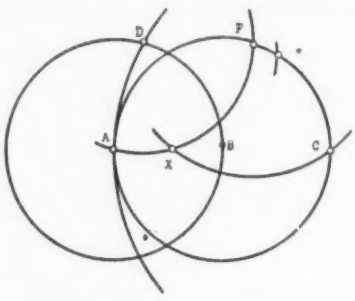


Fig. II

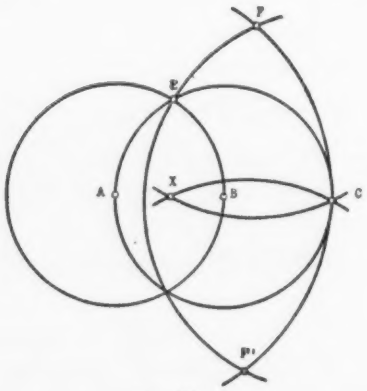


Fig. III

Draw the circles $A(B)$ and $B(A)$. On the second circle, locate the fourth vertex C of the regular hexagon. The points A, B, and C are collinear with $AC=2(AB)$. Let the circle $C(A)$ meet the first circle in D and D'. With these two points as centers and radius AB, construct circles meeting in A and X. This last is the midpoint of AB. For,

$$(AX)/(AD)=(AD)/(AC)$$

and, since $AD=AB$, $AC=2(AB)$,

$$AX=(AB)/2.$$

II. Solution by *D. L. Mackay*, Evander Childs High School, New York.

Locate the point D as in Solution I. Then draw the circle $D(A)$ meeting $B(A)$ in F. The circle $F(C)$ will meet $D(A)$ in X, the midpoint of AB. For, by construction, $AD=DX=BF$, $DC=AC=2(AB)$ and ABC is a straight line. In triangle ADC, we find by the median formula: $DB=(AB)\sqrt{6}/2=FX$. If DX' is the median of triangle ADB, we find $DX'=AB$. Hence DX coincides with DX' since circle $D(A)$ cannot cut AB in more than two points.

III. Second solution by *Johannes Mahrenholz*, Gottbus, Germany.

Let the circles $A(B)$ and $B(A)$ intersect in E and $C(E)$ and $A(C)$ intersect in F and F'. With F and F' as centers draw the two circles through C. Their other intersection locates the midpoint X. For,

$$(XC)/(CD)=(CD)/(AC).$$

But $$CE=CD=(AB)\sqrt{3},$$

and thus $$(XC)(AC)=3(AB)^2,$$

or, since $AC=2(AB)$, $$XC=3(AB)/2.$$

Editor's Note: Solutions I and III are those of *Mascheroni* while II is credited by *MacKay* to *Karl Herbst* in Unterrichtsblätter für Mathematik und Naturwissenschaft, Vol. 15, 1909. Solution III also appears in *H. Steinhaus*, Mathematical Snapshots, Stechert, 1938, p. 38. Solution I is to be preferred since seven arcs only are employed with two different radii. In II there are seven arcs with three radii; in III eight arcs with three radii. Notice that all constructions deal directly with the inverse of the point C in the circle $A(B)$. The construction of an inverse point by means of the compasses alone, as shown in Fig. I, is well known.

Also solved by *L. M. Kelly*, *J. Rosenbaum*, *C. W. Trigg*, and the *Proposer*. Several other constructions were received based upon the determination of points as the intersections of tangent circles. Since, among other reasons, accuracy would be forfeited by such constructions, they were not considered legitimate.

No. 272. Proposed by *Walter B. Clarke*, San Jose, California.

Given the line segments of lengths d, e, f. Locate these as three chords concurrent at P in a circle of minimum size so that P divides each chord as the orthocenter would divide it if they were altitudes of a triangle. What are the sides of the corresponding triangle?

Solution by *D. L. MacKay*, Evander Childs High School, New York.

In triangle ABC, if d, e, f are the respective altitudes to a, b, c, then $a:e=b:d=c:de/f$. Construct triangle ABC similar to triangle $(e,d,de/f)$ and draw its altitudes AD, BE, CF intersecting at H.

Assuming that d is the largest segment, then on any diameter KL in a circle of radius $d/2$, take $KP=AH$. Draw circles $P(BH)$ and $P(CH)$ cutting the first circle at M and N, respectively. Prolong MP and NP to cut the circle again at R and S, respectively. The required chords are KL, MR, and NS.

If T and T' are the areas of triangle ABC and the triangle whose sides are d, e, f, then since the foregoing proportion may be written $a:1/d=b:1/e=c:1/f$, we have

$$T:T'=a^2:1/d^2=a^2d^2=4T^2,$$

whence

$$4TT'=1$$

and

$$a=1/2d\ s'(s'-d)(s'-e)(s'-f),$$

where $s'=(d+e+f)/2$.

No. 274. Proposed by *I. C. Nichols*, Louisiana State University.

Show that the flex point of a cubic is its point of symmetry.

Solution by *Thomas F. Mulcrone*, Spring Hill College, Alabama.

Let the equation of the cubic be:

$$y=x^3+px^2+qx+r. \tag{1}$$

At a flex point the second derivative, y'', is zero. That is,

$$6x+2p=0. \tag{2}$$

Solving this for x and replacing in (1), the coordinates of the flex point, P, are found to be.

$$(x,y)=(-p/3,\ 2p^3/27-pq/3+r).$$

Let $P_1 : (x_1,y_1)$ be any point of the curve and $P_2 : (x_2,y_2)$ on the line PP_1 and such that $PP_1 = P_2P$. That is,

$x = (x_1+x_2)/2 = -p/3,$ or $x_2 = -2p/3 - x_1;$

$y = (y_1+y_2)/2 = 2p^3/27 - pq/3 + r,$ or $y_2 = 4p^3/27 - 2pq/3 + 2r - y_1.$

If these values for x_2 and y_2 be substituted in (1), we have:

$$y_1 = x_1^3 + px_1^2 + qx_1 + r.$$

Hence, if P_1 is on the curve, then P_2 is also on the curve; and so P, the flex point, is the point of symmetry.

Also solved by the *Proposer*.

No. 275. Proposed by *H. T. R. Aude*, Colgate University.

The sum of two dollars and seventy-one cents is made up of twenty-two coins which are pennies, nickels, dimes, and half-dollars. If there is a different number of each kind is the solution unique?

Solution by *T. F. Mulcrone*, Spring Hill College, Alabama.

Let p, n, d, and h be the number of pennies, nickels, dimes and half-dollars, respectively. By hypothesis:

(1) $$p + 5n + 10d + 50h = 271,$$

(2) $$p + n + d + h = 22,$$

(3) there cannot be more than 16 coins of any one kind.

From (1) and (3), p is restricted to the values 1, 6, 11, 16. With $p = 1$, (1) and (2) give $n + 2d + 10h = 54$, $n + d + h = 21$, hence $d + 9h = 33$. Thus $h < 4$. If $h = 3$, then must $d = 6$ and $n = 12$; if $h = 2$, then $d = 15$, $n = 4$; if $h = 1$, then $d = 24$ which must be excluded. Analogous discussion of the other values of p produces immediately the following four solutions: (p, n, d, h) are given by (1, 4, 15, 2), (1, 12, 6, 3), (6, 11, 1, 4), and (11, 2, 5, 4).

Also solved by *C. W. Trigg*.

No. 276. Proposed by *E. P. Starke*, Rutgers University.

Having given an arithmetic progression of real or complex terms with the property that the terms can be rearranged to form a geometric progression, prove that there exists another permutation of the terms which gives a harmonic progression.

Solution by the *Proposer*.

It will be shown that the only progressions that can exist under the hypothesis have their terms proportional to (a) 1, 1, . . . , 1 or (b) 4, 1, −2. Any permutation of the terms of (a) is at once arithmetic, geometric and harmonic; of the six permutations of (b) evidently two give arithmetic, two, geometric and two, harmonic progressions.

Let three consecutive terms of the given arithmetic progression be $2r$, $r+s$, $2s$. The conditions that these be in geometric progression, in the various orders, are

$$(r+s)^2=4rs, \qquad 4r^2=2s(r+s), \qquad 4s^2=2r(r+s).$$

From these we have $r=s$, $s=-2r$, or $r=-2s$. Thus the terms are proportional to (a) or (b) above. The progression (b) evidently cannot be extended another term in either direction, for the additional term, 7 or −5, destroys the possibility of forming a geometric progression.

PROPOSALS

Note: The following problems remain unsolved to date. Readers are urged to submit solutions or discussions.

From Vol. X: 105, 107, 114, 127.
Vol. XI: 132, 133, 134, 150, 155.
Vol. XII: 162, 178, 180, 192, 202, 206, 215, 224, 225, 226, 230, 231, 238.

No. 300. Proposed by *G. W. Wishard*, Norwood, Ohio.

Prove that any odd square except 1 may be formed as the sum of two or three squares of positive integers.

No. 301. Proposed by *V. Thébault*, Le Mans, France.

Determine the system of numeration in which a number of the form *abcabc* is the cube of a number *dd*.

No. 302. Proposed by *Walter B. Clarke*, San Jose, California.

Construct an equilateral triangle whose vertices lie at a given point and upon each of two given lines.

No. 303. Proposed by *H. T. R. Aude*, Colgate University.

Find the smallest quadrilateral with unequal integral sides such that each of two opposite angles is 120°.

No. 304. Proposed by *Yudell Luke*, University of Illinois.

A and B are the two parabolas $y^2=2px$ and $x^2=2qy$, respectively, whose points of intersection are the origin and C. The tangents from C to A and B meet B and A at the points D and E, respectively. Show that DE is a common tangent of A and B.

No. 305. Proposed by *H. S. Grant*, Rutgers University.

There are $n+3$ urns, $U_1, U_2, \ldots, U_{n+3}$. Each of the urns, $U_1, U_2, \ldots, U_{n+1}$ contains a white and b black balls, $n \leqq a+b$; each of the urns U_{n+2}, U_{n+3} contains c white and d black balls. n balls are drawn, one at a time, from U_1 and placed in U_{n+2}; the $c+d+n$ balls in U_{n+2} are then thoroughly mixed. One ball is drawn from each of the n urns $U_2, U_3, \ldots, U_{n+1}$ and placed in U_{n+3}; the $c+d+n$ balls in U_{n+3} are then thoroughly mixed. Show that the probabilities that a ball extracted from U_{n+2} be white or black are the same as the corresponding probabilities for a ball extracted from U_{n+3}, and find simple expressions for these probabilities.

No. 306. Proposed by *C. N. Mills*, Illinois State Normal University.

Without locating the points of contact, find the equation of the conic touching the lines $x=0$, $y=0$, $x=2$, $x-y+1=0$, $x+y-3=0$.

No. 307. Proposed by *E. P. Starke*, Rutgers University.

It is known that there are infinitely many pairs of consecutive squares whose sum is a square. What is the smallest value of $k>2$, such that the sum of the squares of k consecutive positive numbers can be a square?

No. 308. Proposed by *Albert Farnell*, Louisiana State University.

Sum the infinite series whose representative term is $n^2/(n-1)!$

No. 309. Proposed by *W. V. Parker*, Louisiana State University.

If three lines $a_ix+b_iy+c_i=0$ are such that

$$a_1+a_2+a_3=b_1+b_2+b_3=0,$$

show that the area of the triangle formed by them is $|(c_1+c_2+c_3)^2/2\Delta|$, where $\Delta=|a_ib_i|$.

Corrections: (March issue)

p. 285, line 5 from bottom: put x^3 for x^2.
p. 286, line 7 from bottom: delete "which".
p. 292, line 17: delete the prime in ${}_nS_{j'}$.
p. 292, line 17: read ${}_nS_{j'}-{}_nS_j$.

Bibliography and Reviews

Edited by
P. K. SMITH and H. A. SIMMONS

CORRECTION

(Misprint in April, 1939 issue N. M. M. read 175 pages.)

Introduction to the Theory of Groups of Finite Order. By Robert D. Carmichael. Ginn and Company, Boston, 1937. xiv+447 pp. $5.00.

Theoretical Mechanics—A Vectorial Treatment. By C. J. Coe, The Macmillan Company, New York, 1938. vii+554 pages. Price $6.00.

In the Preface to *Theoretical Mechanics*, Professor Coe (assistant professor of mathematics at the University of Michigan) makes the following observations concerning the text, clearly defining its nature and purpose:

"The study of Theoretical Mechanics has undergone profound changes in recent years both in the formulation of the fundamental principles on which it rests and in the analytic procedure by which its conclusions are derived. ... The present point of view may perhaps be expressed by saying that we regard geometry, kinematics, classical mechanics, relativity mechanics as different theories, each resting on its own postulates, each consistent in itself and each possessing certain advantages in its degree of simplicity and in the degree of accuracy with which it interprets the physical world. ... The concept of *vector* is, of course, inherent in *mechanics* and must appear in any exposition of it, but the special methods and notation of *vector analysis* have only recently come into general use in the study of *mechanics*.

"It has been the author's aim in preparing this book to present an introduction to classical mechanics, and through it to mathematical physics, which shall make clear the simple postulates on which they rest and at the same time to explain and apply immediately the basic principles of vector analysis. The student learns vector analysis almost incidentally in the study of mechanics, and learns it well because he not only studies it directly but continues to use it thereafter".

The table of contents, briefly annotated by the reviewer, is as follows: 1. Introduction (includes postulates and undefined elements for mechanics); 2. Rectilinear Motion of a Particle (includes a careful discussion of scalar quantities); 3. Introduction to Vectors; 4. Curvilinear Motion of a Particle (includes projectile motion, pendulum motion, and planetary motion, all excellently portrayed); 5. Displacements and Motion of a Rigid Body; 6. Sliding Vectors; 7. Statics of

Particles and of a Rigid Body; 8. Statics of a Flexible Cord; 9. Principle of Virtual Work; 10. Kinetics of Particles (with an interesting short discussion of the Problem of N-Bodies); 11. Kinetics of a Rigid Body (includes a discussion of the Motion of a Top); 12. General Principles of Mechanics (D'Alembert's Principle, Lagrange's Equations, Hamilton's Canonical Equations, Hamilton's Principle, Principle of Least Action, Gauss' Principle of Least Constraint); 13. Vector Calculus; 14. Potential Theory.

The book is designed for a two year course, introductory and advanced, but could be used for a shorter course by a selection of topics. The text, with the exception of the last three chapters, requires no more than a year of calculus as prerequisite. The last chapters require an elementary course in differential equations.

There is need for such a course as this for undergraduates who are interested in mathematical physics, and for those graduate students who do research work in that topic. A course based on Professor Coe's text would be admirable for advanced undergraduate and first year graduate students in mathematics, for it would not only teach vector methods and mechanics but would also afford an excellent introduction to the modern postulational approach in mathematics. It would, moreover, assist in developing ability in analyzation. The method of presentation is through the medium of theorems which are carefully stated and rigorously proved. Care is used to indicate the postulates and undefined elements. The text contains a wealth of problems which serve to drill the student in the topics at hand and to develop additional theorems—some of historical interest. Vector methods are advantageously used throughout, and the reviewer believes that a student would have a working knowledge of these after a first course with this text. In particular, he would have an excellent background for a study of modern differential geometry.

The reviewer would prefer to see a few more free-body diagrams for the illustrative problems of the text and a more general inclusion of units with numerical answers.

North Carolina State College. JOHN W. CELL.

Der deutsche Verein zur Förderung des mathematischen und naturwissenschaftlichen Unterrichts e.V., 1891-1938. By Wilhelm Lorey. Verlag Otto Salle, Frankfurt am Main, 1938. 165 pages, with 28 illustrations. Paper cover; 3 marks.

This splendid volume gives a very complete history of the German "Förderverein" (association for the advancement of instruction in

mathematics and science) from its founding in 1891 up to April 11, 1938, when it was reorganized and its name changed. The executive committee could scarcely have chosen a better qualified man to write the organization's history; Professor Lorey has been attending meetings of the society since 1895, is one of the leading mathematical historians, and reports these happenings from first-hand knowledge. His work, as usual, bears the imprint of broad interests, genuine scholarly spirit, and careful preparation. Although there is a great mass of information and attention to detail, this reviewer found the account interesting and readable. The inevitable ups and downs of such an organization are not missing here.

It is striking to note the opposition from those representing the humanities which had to be overcome in Germany in advancing curricular and methodological reforms for teaching mathematics and science. The great work of Felix Klein along this line receives full treatment. His executive ability and talent in organizing were an important factor. Professor Lorey has also included a history of the *Unterrichtsblätter für Mathematik und Naturwissenschaften*, the official organ of the association. In one sense it might well be compared to the *Mathematics Teacher* and several other American journals.

Many efforts of the society centered about the Meran Curriculum (1905), which is still important today in its effects. Another great undertaking supported by the Förderverein was the German mathematical encyclopædia begun in 1898 and finished in 1935.

Professor Lorey gives much pertinent information about the founding of the International Commission on the Teaching of Mathematics and the part played by his fellow countrymen. He traces the effects of the world war and the inflation period on the teaching of mathematics and science, and discusses new problems arising in contemporary Germany.

The notes are extremely complete and contain a wealth of historical information and sketches of scientists as well as mathematicians. The format and typography leave nothing to be desired. The work is quite free of misprints. There are two extensive indexes, important in a work of this kind, one of topics and one of proper names. The portrait illustrations of outstanding men such as Felix Klein, Kerschensteiner, Lietzmann, H. Wagner, and Gutzmer add much to the attractiveness and value of the volume. Under each is a facsimile autograph.

State Teachers College
La Crosse, Wisconsin.

G. WALDO DUNNINGTON.

Higher Mathematics. By R. S. Burington and C. C. Torrance. McGraw-Hill Book Company, Inc., New York, 1939. xiii+844 pages.

This book is devoted to an exposition of advanced calculus for students of physics and engineering. The nine chapters cover, besides some topics of a first course in calculus, line integrals, surface integrals, volume integrals, Jacobians, improper integrals, integrals depending on a parameter, numerical integration, ordinary differential equations with applications to electrical networks, series, Fourier series, complex variables including elliptic functions, the elements of the theory of matrices, an introduction to vector analysis with applications to differential geometry, tensors, partial differential equations and applications to mathematical physics, calculus of variations, analytical dynamics, and an introduction to the theory of real variables. A bibliography of over four pages appears at the end of the book.

Misprints were noted on pages 303, 367, 527, 583, 613. On pages 56-59 the distinction between dx/dt, ds/dt and speed is not maintained throughout. On page 62 curvature is introduced under the assumption that $ds/dx>0$, whereas the orientation of the arc may be given also by $ds/dx<0$. On page 191 the trigonometric substitutions used require a specification of definite inverse functions. In three dimensional figures left-handed coordinate axes prevail. Some of the notations used in integration are unusual, to say the least. Matrix theory and infinite series are used in the chapter on differential equations before a discussion of them appears in later chapters.

The text is provided with excellent figures and the printing is very good. The presentation of material, especially the applications to science, is both clear and careful. Interesting problems abound. The book should become a useful reference for students and instructors in applied mathematics.

Virginia Military Institute. W. E. BYRNE.

Enriched Teaching of Mathematics in the Junior and Senior High School. Revised edition. By Maxie Nave Woodring and Vera Sanford. Bureau of Publications, Teachers College, Columbia University, New York, 1938. ix+133 pages.

A good number of readers of this journal are unacquainted likely with the *Enriched Teaching Series.* This series includes descriptions of reference material, in separate publications, on commercial subjects, English, Latin, mathematics, physical education, and science. Reference works in other subjects are in preparation. The moving spirit in

these publications is Prof. Maxie Nave Woodring, of Columbia University. Professor Maxie's name appears on each text as one of two authors.

If the other subjects are as well done as the mathematics, the reviewer testifies, without fear of exaggeration, that the series is a most valuable contribution toward the improvement of teaching in these fields. The work on mathematics has to do primarily with materials in junior and senior high school. Increasingly, we are challenged as college teachers to vitalize our subject matter. This reference material may be used also with profit in the enrichment of elementary college mathematics.

The reader will be interested in a listing of the topics given. References are given on units of work, standards tests, and work books in arithmetic, algebra, intuitive and demonstrative geometry, and trigonometry; on class room equipment, pictures, lantern slides, and exhibits; on periodicals, yearbooks, and the history of mathematics. The text closes with sixty-one addresses of sources and an index. With each reference is given the publisher or manufacturer, the date of publication, and usually, the price. Every teacher of high school and college freshman mathematics should have easy access to this text.

Louisiana Polytechnic Institute. P. K. SMITH.

Members of the Editorial Board of this journal acknowledge with appreciation the assistance of the following mathematicians, who have acted as referees for papers submitted for publication in National Mathematics Magazine, throughout the current year, Volume XIII:

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$$\frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)}\,x^{p-1}(1-x)^{q-1}$$

A Problem in Minimum Values

The Poristic Locus of the Gergonne Point

An Overlooked Numeral System of Antiquity

An Approach to Trigonometry

Mathematical World News

Problem Department

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